On Certain Unicursal Twisted Curves.

By Virgil Snyder.

1. When the tangents to a twisted curve belong to a linear complex, the curve must be self dual. If it be of order n, the class of the osculating developable is also n. Let P be any point not on the curve c_n. From P can be drawn n osculating planes, the points of osculation being $P_1, P_2 \ldots . P_n$. Since PP_i passes through P_i and lies in the polar plane of P_i as to the complex (osculating plane of c_n at P_i) it belongs to the complex, hence the polar plane of P contains $P_1 \ldots . P_n$. *If through any point the osculating planes be drawn to a twisted curve contained in a linear complex, the points of osculation will lie in a plane passing through the given point.*

2. Let c_n be projected into a plane curve from P. The n osculating planes from P will cut the plane of the projected curve in inflexional tangents, and the points of inflexion will be in a straight line. *When a plane curve $\phi_n = 0$ is the projection of a twisted curve of the same order which belongs to a linear complex it must have n points of inflexion on a straight line.* This necessary condition is not sufficient, as $\phi_4 = 0$ exist having four collinear inflexions, and no singular points, but a twisted quartic of genus 3 does not exist.

For a unicursal c_n however, the condition is easily proved to be necessary and sufficient. Such a curve ϕ_n has $3(n-2)$ inflexions. The $2(n-3)$ inflexions of ϕ_n which are not included among the n collinear ones, are also inflexions on the twisted c_n. In general, the tangents at the points of a twisted curve at which the osculating planes have four point contact are ordinary, but in this case the stationary planes coincide in pairs, forming $2(n-3)$ points of linear inflexion.*

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* The theorem of § 1 was proved analytically for $n = 4$ by Appell, *Grunert's Archiv*, vol. 62, p. 175 (1878). The second theorem was proved analytically for unicursal curves and a particular projection by Picard in *Annales de l'Ecole Normale* (1877).

A unicursal twisted curve of order n which belongs to a linear complex and has no point singularities has $2(n-3)$ *points at which the tangents have three point contact.* It has no stationary planes except those containing the inflexional tangents.

3. The points of c_n in the osculating planes form a symmetric $(n-3)$ correspondence which is in general not an involution. The branch points of the correspondence are the points of tangency of the inflexional tangents. Hence it follows that if the osculating plane at A cuts c_n again at B, then the osculating plane at B will contain the point A. The lines joining the points of osculation to each of the $n-3$ residual points of the curve in the plane are therefore principal secants. In particular, if $n=4$, we see that the quartic belonging to a linear complex has an infinite number of principal secants. The general unicursal quartic has three, and they meet in one point; in the present case no three can intersect in one point, for they all belong to a linear complex, and the polar plane of the point would have to cut the quartic in six points.

4. Two kinds of curves will now be discussed, one having a pair of maximum line singularities, the other having a pair of maximum point singularities.

Consider the curve

$$x:y:z:w=\lambda^n:n\lambda^{n-1}:n\lambda:1. \tag{1}$$

It lies on the R_2 whose equation is $n^2xw=yz$, and has $x=0$, $y=0$; $z=0$, $w=0$ for tangents having n — 1 points of contact, the former at $(0,0,0,1)$, the latter at $(1,0,0,0)$. These points correspond to the values $0, \infty$, of λ. The equation of the osculating plane at λ is

$$(n-2)(\lambda^n w-x)+\lambda^y-\lambda^{n-1}z=0. \tag{2}$$

The plane (2) will meet c_n in points μ defined by

$$(n-2)(\lambda^n-\mu^n)-n\lambda\mu(\lambda^{n-2}-\mu^{n-2})=0, \tag{3}$$

which has the factor $(\lambda-\mu)^3$. The residual factor is a homogeneous symmetric correspondence between λ and μ, which is therefore composite. When n is an even integer, $\lambda+\mu$ is also a factor, each of the $\frac{n}{2}-2$ remaining factors being of the form

$$(\lambda+\mu)^2+\alpha_i\lambda\mu. \qquad \left[i=1,2\ldots\frac{n}{2}-2.\right] \tag{4}$$

When n is odd, each of the $\frac{n-3}{2}$ factors is of this form. If the ruled surface be constructed by connecting the point λ to each of the $n-3$ points μ, it will

break up into factors. The expression (4) equated to zero is also factorable, but each factor will define the same scroll. The scrolls arising from the different factors of the form (4) are entirely distinct.

5. When n is even, the lines joining each point λ to $-\lambda$ form a surface on which c_n is a simple curve. The points $\lambda, -\lambda$ form a quadratic involution having 0, ∞ for double elements. The equation of the scroll is

$$x^{\frac{n}{2}-1} z^{\frac{n}{2}} = y^{\frac{n}{2}} \cdot w^{\frac{n}{2}-1}, \tag{5}$$

i. e. a scroll of order $n-1$, having $y=0$, $z=0$, the line joining the points of inflexion, for $\frac{n}{2}$-fold directrix, and $x=0$, $w=0$, the intersection of the stationary planes, for $\left(\frac{n}{2}-1\right)$-fold directrix. The two directrices are polar lines with regard to R_2 on which c_n lies, and also with regard to the complex to which it belongs. The lines $x=0$, $y=0$ and $z=0$, $w=0$ are each $\left(\frac{n}{2}-1\right)$ fold generators. There are no pinch points on $x=0$, $w=0$, and the only ones on $y=0$, $z=0$ are at $(1, 0, 0, 0)$, $(0, 0, 0, 1)$. The osculating planes at λ and at $-\lambda$ form with the planes through λ and the directrices of the scroll a harmonic pencil. A general plane section has an $\frac{n}{2}$-fold point on one directrix, and an $\left(\frac{n}{2}-1\right)$-fold point on the other directrix and on each inflexional tangent. The singularity on the latter lines has the defining terms $\zeta^{\frac{n}{2}} = x \cdot \xi^{\frac{n}{2}-1}$, hence it counts for $\frac{n}{2} - 2$ cusps. There are no other point singularities, hence the class is n. Finally, it is evident that c_n is an asymptotic line on the surface, since a generator at λ always lies in the osculating plane of λ. It is the lowest order that a scroll of order $n-1$, contained in a linear congruence can have, when each generator is a bisecant of the curve.

All the asymptotic lines on the scroll (5) are of the same form as c_n, each having the same lines for $n-1$ point tangents at the same points. For $n=4$ we have the most general c_4 contained in a linear complex. Two orthogonal projections of a series of such curves, as asymptotic lines on a cubic scroll, are drawn in the Bulletin of the American Math. Soc. vol. 5 (1899), p. 349.

6. The other scrolls are all of the same nature, all being defined by (4). The two values of μ corresponding to a given value of λ are of the form $\varkappa\lambda$ and $\frac{\lambda}{\varkappa}$ If we consider a projective relation τ between λ, μ, having $0, \infty$ for selfcorresponding elements, the equations will be of this form. Associated with every point λ are two points $\tau\lambda$ and μ such that $\tau\mu = \lambda$, hence the two factors $(\mu - \varkappa\lambda)(\varkappa\mu - \lambda)$ will define the same scroll. Since two generators issue from each point λ of c_n, the curve is a double curve on the surface. If in (1) we write the equations of a line joining λ to $\varkappa\lambda$, the result will be a scroll of order $2(n-1)$. Moreover, in the present case both generators lie in a plane containing the tangent, hence c_n is tacnodal, and every point of it is a uniplanar point on the surface, instead of a biplanar point, as in case of an ordinary double curve. Moreover, all the generators lie in the osculating planes of c_n, hence the curve is an asymptotic line on the surface, and indeed for each of the two branches. From the equations of a generator it appears that the inflexional tangents are each $\frac{1}{2}(n-3)(n-2)$-fold lines on the surface. If $\varkappa$ is an $(n-1)$ root of unity, the $n-1$ points $\lambda, \varkappa\lambda, \varkappa^2\lambda \ldots$ are all collinear. The surface is now R_2.

7. Finally, if a generator be defined by the value t of λ, and another be defined by s, the condition that the generators s and t intersect is a symmetric correspondence between s, t. Let $s + t = \sigma$ and $st = \tau$. The equation $F(\sigma, \tau) = 0$ becomes in our case factorable, one factor of the form (4) being squared, and $2n - 4$ other factors of the form $\sigma^2 + \beta_i\tau$. From the theorem recently proved by Mr. Sisam it follows that each nodal factor is a unicursal curve of order n, having each generator as a bisecant, and the points of intersection define a projectivity having $0, \infty$ for selfcorresponding elements. Hence every component nodal curve passes through both of the points of inflexion.

8. The curve (1) will project into a plane curve having two tangents of $(n-1)$ points of contact, and n collinear points of inflexion. For points on the line joining $0, \infty$, the two former points coincide at a node. For $n = 4$, the lemniscate belongs to this type.

9. The curve

$$x : y : z : w = \lambda^{2n+1} : \lambda^{n+1} : \lambda^{n-1} : 1 \tag{6}$$

has a similar property. It belongs to a linear complex and lies on a quadric surface. It has two n-fold points, all the tangents at each being coincident. The locus of the lines joining a point λ to the $2n-2$ points μ in the osculating plane breaks up into $n-1$ surfaces, each one rational, of order $2(n+1)$, and

having the original curve for a tacnodal component of the nodal curve. The scroll has no multiple generators. The residual nodal curve on each component consists of $n-2$ other curves, each one rational, each of order $2n+1$, each cut by every generator twice. The points of intersection of a generator with each component determine a projectivity having the singular points $0, \infty$ for self-corresponding elements.

The condition for a tacnodal curve is also satisfied when $\mu = \alpha\lambda$, α being a $(n+1)$th root of unity. The points $\lambda, \alpha\lambda, \alpha^2\lambda, \ldots$ are now collinear; they are the points of intersection of the curve with the generators of the R_2 on which the curve lies. The points form an involution of order $n+1$, and R_2 is counted $(n+1)$ times. As in the preceding case, every point of (6) is a uniplanar point on each of the scrolls, and the curve is an asymptotic line for both branches. The simplest case for which these scrolls have a meaning is when $n=2$. The equations of a generator become

$$\lambda^3 z + 2\lambda^2 y - x = 0, \quad \lambda^3 w - 2\lambda z + y = 0,$$

and the equation of the scroll is

$$\begin{vmatrix} 2yw & 2z^2 & zy + xw \\ 2z^2 & 3yz - xw & 2y^2 \\ zy + xw & 2y^2 & 2xz \end{vmatrix} = 0. \qquad (7)$$

An arbitrary plane will cut a sextic curve from this surface, having five tacnodes; a plane containing a generator will cut a quintic curve having three tacnodes and having the generator for bitangent. A plane containing two generators (osculating plane of c_5 at λ) will have a cusp at λ and a node at each of the residual points μ. The scroll (7) is not included in Wiman's list of sextic scrolls. He overlooked the entire possibility of such configurations. It is my number 50, but the equation is not derived.* A plane projection of (6) will be a rational c_{2n+1} with two points that are n-fold, all the tangents coinciding. It also has n^2 double points, and $2n+1$ points of inflexion, all collinear. When the center of projection is taken on the line joining the two cusps the apparent double points all coincide with the cusps, the tangents at which are in different directions. When the cusps are real, part of the points of inflexion must be

* JOURNAL, vol. 27 (1905), p. 188.

imaginary. The simplest curve of this type is the nodal cubic. For c_5, compare types 143–145 of Field's enumeration of unicursal quintics.*

10. The curve

$$x : y : z : w = \lambda^n : n\lambda^{n-1} : \binom{n}{2} \lambda^{n-2} : 1 \tag{8}$$

lies on an R_2, has an $(n-1)$-fold point at (0, 0, 0, 1) with coincident tangents, and a stationary plane having n-point contact at (0, 0, 0, 0), the plane being $w = 0$. The curve does not belong to a linear complex, but has some properties analogous to the preceding ones. The lines joining λ to the $n-3$ points μ generate $n-3$ different scrolls, each of order $n+1$, and of type $[2, n-1]$, *i. e.* it is generated by the line of intersection of a tangent plane to a quadric cone and that of a planar developable of class $n-1$, when the planes are in (1, 1) correspondence. The curve (8) is a double (not tacnodal) curve on each, but is an asymptotic line on one of the branches. When n is even, $n = 2m$, c_n lies on a cone $\varkappa_m$, $x^{m-1} w = z^m$, forming the complete intersection of $\varkappa_m$ and $\varkappa_2$, $y^2 = xz$. The nodal curve of each R_{n+1} is of the same form for each component scroll; it consists of $m-2$ distinct curves besides (8), each one having the generators for bisecants. A similar projectivity ($\mu = \varkappa\lambda$) is defined on each component double curve. The line joining λ to its harmonic conjugate with regard to $0, \infty$ (cusp, stationary plane) will generate $\varkappa_m$. The factor corresponding to it will always appear in the criterion for the factorability of the nodal curve. The simple factor defines another nodal curve which is cut but once by each generator. The order of this curve is m. When n is odd, the harmonic conjugate does not lie in the osculating plane of λ. For $n = 5$, the scrolls are of type 19 in my list (l. c. p. 177).

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* P. Field, "On the Forms of Unicursal Quintic Curves," JOURNAL, vol. 26 (1904), p. 149.